Eigenmodes of decay and fragmentation processes

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Linear rate equations are used to describe the cascading decay of, e.g., a jet into partons or an initial heavy nucleus into fragments. This representation is based upon a triangular matrix of transition rates. We expand the state vector of mass multiplicities, which describes the process into the biorthonormal basis of eigenmodes provided by the triangular matrix. We obtain analytic solutions of *discrete* models with explicit mathematical properties for the eigenmodes. A suitable continuous limit, valid for large systems, provides a solution interpolating between the solvable discrete cases. It gives a general relationship between the decay products and the elementary transition rates.

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For the analysis of fragmentation and decay processes it is possible to consider binary [1] elementary processes where a suitable observable (mass number, energy momentum, etc.) with value k breaks into parts with values j and $k-j$, $j=1,2,\ldots,k-1$, with a probability w_{jk} per unit of time. For simplicity, we assume that w_{jk} is time dependent. Also, for clarity, we speak of "mass" and "fragments" in the following. Note that the formalism may as well consider discrete or, at the limit of large "masses," continuous degrees of freedom.

By definition, $w_{jk}=0$ if $j\geq k$ and w_{jk} is symmetric if j is replaced by $k-j$, naturally. (For technical reasons, the coefficient $w_{j,2j}$ is twice the actual transition rate.) Let $N_j(t)$ be the multiplicity of fragment j at time t in a process initiated from the decay of a cluster A, namely $N_j(0)=\delta_{jA}$. The model under study is described by the following set of linear, first order differential equations,

$$\frac{dN_j}{dt}=-c_jN_j+\sum_{k=j+1}^{A}w_{jk}N_k,$$

$$j=1,\ldots,A, \quad c_j=\sum_{l=1}^{j-1}\frac{w_{lj}}{2}, \quad (1)$$

With components N_j, $j=1,\ldots,A$, for a column vector $|\mathcal{N}\rangle$, the system, Eqs. (1), boils down to $d|\mathcal{N}\rangle/dt=\mathcal{W}|\mathcal{N}\rangle$ with a triangular matrix $\mathcal{W}$. The solution of Eqs. (1) is obviously $|\mathcal{N}\rangle=\exp(-t\mathcal{W})|\mathcal{N}_0\rangle$, a sum of exponentials whose rates of decay in time are the trivial eigenvalues of the triangular $\mathcal{W}$, namely the diagonal matrix elements $-c_\lambda$. The purpose of this paper is to take advantage of the expansion of the evolution on the biorthonormal basis of eigenmodes of $\mathcal{W}$,

$$|\mathcal{N}\rangle=\sum_{\lambda=1}^{A}|\lambda\rangle\exp(-tc_\lambda)\langle\bar{\lambda}|\mathcal{N}_0\rangle=\sum_{\lambda=1}^{A}|\lambda\rangle\exp(-c_\lambda t)Y_A^\lambda,$$

$$(2)$$

where Y_A^λ is the last component of the bra eigenvector $\langle\bar{\lambda}|$. We are specially interested in the analytical structure of the

bra eigenvectors $\langle\bar{\lambda}|$, because of the synthetic information carried by scalar products $M_\lambda=\langle\bar{\lambda}|\mathcal{N}\rangle$.

In our previous investigations [1] we found numerical evidence for specific exponents $q(\lambda)$ which govern a powerlike increase of the components of the bra eigenvectors, $Y_j^\lambda\propto j^{q(\lambda)}$. In other words, M_λ is similar to a moment of order $q(\lambda)$ of the mass distribution, $M_\lambda\simeq\Sigma_j j^{q(\lambda)}N_j$. We took advantage of the fact that $\mathcal{W}$ has a fixed bra eigenvector $\langle\bar{1}|$, whose components $Y_j^1=j$, $j=1,\ldots,A$, express mass conservation. Indeed the total mass is $M_1=\langle\bar{1}|\mathcal{N}\rangle=\Sigma_{j=1}^{A}jN_j$, with $dM_1/dt=\langle\bar{1}|\mathcal{W}|\mathcal{N}\rangle=0$, since the first eigenvalue $-c_1$ identically vanishes. It was also convenient to define the "mass weighted multiplicity" (MWM) vector U with components $U_i=iN_i$, whose evolution is governed by a matrix $\mathcal{V}$, with matrix elements $\mathcal{V}_{jk}=j\mathcal{W}_{jk}/k$, hence $dU/dt=\mathcal{V}U$. A numerical analysis of U revealed scaling properties of the fragmentation process. The present paper attempts to give analytical proofs where only numerical evidence was previously found. In particular we are interested in those mathematical properties of the bra $\langle\bar{\lambda}|$ eigenvectors which may enlighten the comprehension of the expansion, Eq. (2).

Our main tools in this analysis are the following equations for the components Y_i^λ of the bra eigenvectors of $\mathcal{W}$:

$$\sum_{i=\lambda}^{j-2}Y_i^\lambda w_{ij}+Y_{j-1}^\lambda w_{j-1,j}=(c_j-c_\lambda)Y_j^\lambda, \quad j\geq\lambda. \quad (3)$$

Similar equations govern the components X_i^λ of the ket eigenvectors, naturally. We keep in mind that $Y_j^\lambda=0$ if $j<\lambda$ and $X_i^\lambda=0$ if $i>\lambda$. A suitable (biortho)normalization results from the following boundary conditions:

$$Y_\lambda^\lambda=X_\lambda^\lambda=1. \quad (4)$$

Note that the components of the biorthonormal eigenvectors of the matrix $\mathcal{V}$ which governs the MWM are easily deduced, as $\mathcal{Y}_j^\lambda=\lambda Y_j^\lambda/j$ and $\mathcal{X}_i^\lambda=iX_i^\lambda/\lambda$.

It happens frequently that c_j is a monotonically increasing function of j. The physical interpretation is that the heavier the system, the larger its total decay rate. Under such a condition, the coefficients $c_j - c_\lambda$ on the right-hand side of Eqs. (3) are always positive when $j > \lambda$. The left-hand sides of the same equation, Eqs. (3), depends only on the positive coefficients w_{ij} and components Y_i^λ of rank i lower than j, $\lambda \leq i \leq j - 1$. It is easy to calculate $Y_{\lambda+1}^\lambda$ from Y_λ^λ alone, which was chosen to be positive from the normalization $Y_\lambda^\lambda = 1$. Thus $Y_{\lambda+1}^\lambda$ is positive. A trivial recursive argument then shows that *all* nonvanishing components Y_j^λ are positive. As a consequence of the positivity of the bras and of biorthogonality, the ket eigenvectors must show highly oscillatory components. This demands a further, specific study.

In the same way as in our earlier [1] studies we first select values of w_{ij} which express scaling properties at vertices of binary fragmentation. For this, the family of models we consider [2] uses the form $w_{ij} = [f(i/j) + f(1 - i/j)]j^{-a}$, and is parametrized by a "splitting" function f and an overall exponent a. For most of the present analytical study, we restrict a to the value $a = 1$, because it allows a convenient continuous limit, valid for finite values of λ and large sizes A of the matrix. The value $a = 1$, for instance, is the proper exponent for quantum chromodynamics fragmentation [3,4] when we consider energy-momentum fragmentation rather than mass fragmentation.

A simplified, albeit generic form of f is parameterized by a second exponent, b, in the form

$$w_{ij} = \left[\left(\frac{i}{j}\right)^{-b} + \left(1 - \frac{i}{j}\right)^{-b}\right]j^{-a}. \tag{5}$$

The variable $x = i/j$ (with its mandatory symmetry complement $1 - x$) is obviously the scaling mass ratio at the fragmentation vertex. We note that the cases $a = 1$, $b = 0$ and $a = 1$, $b = -1$ generate the same eigenvectors, since the corresponding matrices are strictly proportional, namely $w_{ij}|_{b=0} = 2w_{ij}|_{b=-1} = 2/j$.

For $a = 1$, $b = -1$, the simple nature of w_{ij}, namely $w_{ij} = 1/j$, reduces Eqs. (3) into

$$\frac{j-1}{j}(c_{j-1} - c_\lambda)Y_{j-1}^\lambda + \frac{Y_{j-1}^\lambda}{j} = (c_j - c_\lambda)Y_j^\lambda. \tag{3'}$$

Hence a recursion relation for the bra eigenvector components is found,

$$\frac{Y_i^\lambda}{Y_{i-1}^\lambda} = \frac{i + \lambda - 1}{i - \lambda}, \tag{6}$$

and accordingly the bra eigenvector components are given by the formula,

$$Y_i^\lambda = \frac{(i + \lambda - 1)!}{(i - \lambda)!(2\lambda - 1)!}. \tag{7}$$

For large values of j, the recursion relation expands in powers of $1/j$, when j is large, according to

$$\frac{Y_{i+1}^\lambda}{Y_i^\lambda} = 1 + \frac{2\lambda - 1}{j} + O(1/j^2), \tag{8}$$

which means that $Y_i^\lambda \propto i^{2\lambda - 1}$. This proves analytically that the scalar products $\langle \lambda | \mathcal{N} \rangle = \Sigma_{i=\lambda}^A Y_i^\lambda N_i$ of the bra eigenvectors with $\mathcal{N}$ express moments, of order $q(\lambda) = 2\lambda - 1$, of the multiplicity distributions, restricted to fragments at least as heavy as λ. This justifies analytically, for $a = 1$ and $b = -1, 0$, the numerical evidence obtained in our previous paper [1].

For $a = 1$, $b = 1$ the rates become $w_{ij} = j/[i(j - i)]$. The matrix $\mathcal{V}$ is simpler than $\mathcal{W}$, since then $\mathcal{V}_{ij} = 1/(j - i)$ in the upper-right triangle, which does not vanish. The diagonal is unchanged, and so is, naturally, the lower-left vanishing triangle. A straightforward but slightly tedious argument shows that

$$\mathcal{Y}_j^\lambda = \lambda Y_j^\lambda / j = \frac{(j-1)!}{(j-\lambda)!(\lambda-1)!}. \tag{9}$$

One deduces from Eq. (9) that

$$\frac{Y_{j+1}^\lambda}{Y_j^\lambda} = \frac{(j+1)}{(j+1-\lambda)} = 1 + \frac{\lambda}{j} + O(1/j^2). \tag{10}$$

Hence, now, $q(\lambda) = \lambda$. This makes a second case where the power law $Y_j^\lambda \propto j^{q(\lambda)}$ can be proved analytically.

If a is frozen to the value $a = 1$, it is tempting to interpolate between the behavior $q(\lambda) = 2\lambda - 1$ found for $b = -1$ and that one, $q(\lambda) = \lambda$, valid for $b = 1$. Because of the identity of eigenvectors for the $b = -1$ and $b = 0$ cases, a parabolic interpolation

$$q_{\text{int}}(\lambda) = \frac{\lambda(17 - 4b'^2) + 4b'^2 - 9}{8}, \quad b' \equiv b + \tfrac{1}{2} \tag{11}$$

is found.

This interpolation turns out to be consistent with the continuous limit condition considered earlier, see Eq. (9) in Ref. [1],

$$c_\lambda = \frac{\lambda^{b-1}\lambda^{-1}}{2}\sum_{i=1}^{\lambda-1}[i^{-b} + (\lambda - i)^{-b}]$$

$$= \lambda^{b-1}\sum_{i=1}^{\lambda-1} i^{-b}$$

$$= \frac{1}{2}\int_0^1 dx[1 - x^q - (1-x)^q][x^{-b} + (1-x)^{-b}]$$

$$= \int_0^1 dx[1 - x^q - (1-x)^q]x^{-b}, \tag{12}$$

where we take advantage of the symmetry $w_{ij} = w_{j-i,j}$. We can use this continuous limit, Eq. (12), to estimate q as a function of λ when c_λ is known, whether analytically or numerically. The identity of the cases $b = 0$ and $b = -1$ for Eq. (12) is transparent. Also, for any value of b, the pair $\lambda = 1$, $q = 1$ solves obviously Eq. (12). Then, for $b = 0$ one gets at once from Eq. (12) the condition $(\lambda - 1)/\lambda = 1 - 2/(q + 1)$, hence $q(\lambda) = 2\lambda - 1$, in agreement with Eq. (8). Furthermore for $b = 1$, with the ansatz $q(\lambda) = \lambda$, the recursive relation, derived from Eq. (12),
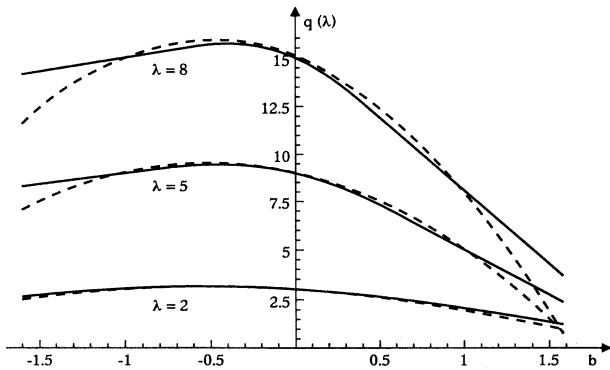
FIG. 1. Behavior of the "eigenexponents" $q(\lambda)$ for the 2nd, 5th, and 8th bra eigenvectors, as functions of the parameter b (model with scaling properties for fragmentation vertices). Full lines: continuous limit predictions; dotted lines: quadratic interpolations between the exact values, known analytically for $b = -1,0,1$.

$$c_{\lambda+1} - c_\lambda = \frac{1}{\lambda} = \int_0^1 dx \, \frac{x^\lambda + (1-x)^\lambda - x^{\lambda+1} - (1-x)^{\lambda+1}}{x},$$

is easy to verify.

We show in Fig. 1 the plots (full lines) of the values of $q(2)$, $q(5)$, and $q(8)$, respectively, obtained numerically from Eq. (12) for $-1.5 \leq b \leq 1.5$. The same Fig. 1 shows (dotted lines) the corresponding interpolation predictions $q_{int}(\lambda)$ from Eq. (11). As long as $-1.2 \leq b \leq 1.2$, the continuous limit results and the parabolic interpolations do not differ by more than 4%. Significant deviations between such estimates occur, however, when $1.2 \leq |b|$. It may be pointed out that the continuous limit estimates are expected to be more rigorous. Finally, in Fig. 2, we show for, e.g., $b = -0.5$, $b = 0.5$, and $b = 1.5$, that, when provided by Eq. (12), the behavior of q as a function of λ remains essentially linear. Namely, a parametrization $q(\lambda) \simeq \alpha(b)\lambda + \beta(b)$ is reasonable, despite the fact that $\alpha(b)$ and $\beta(b)$ are more complicated than the quadratic forms present in Eq. (11).

In conclusion, there is firm ground for the validity of the power law which governs the bra eigenvector components
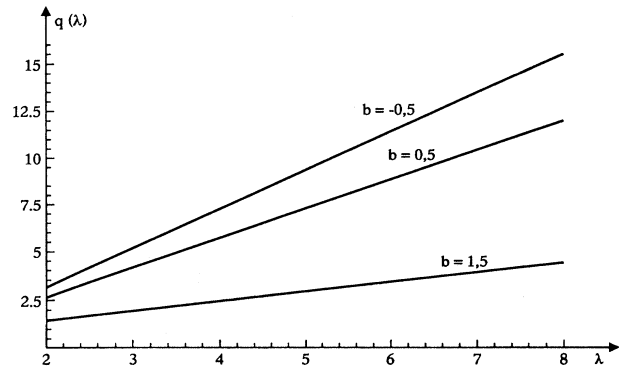

FIG. 2. Strongly linear behavior of $q(\lambda)$ as a function of λ, for different values of b. Full lines are drawn to guide the eye between integer values of λ.

and the corresponding interpretation in terms of moments of the multiplicity vector $\mathcal{N}$. The apparent simplicity of the relation $q(\lambda) \simeq \alpha(b)\lambda + \beta(b)$, see for instance, Eq. (11), expresses a nontrivial relationship between very different quantities, namely, experimentally observable moments on one hand and, on the other hand, much less observable parameters b of the microscopic mechanism of binary fragmentation. This suggests the following criterion for deducing b from experimental multiplicity spectra: given one such spectrum, which is assumed to be parametrized by a freeze-out time t, any ratio $\ln[M_{\lambda'}(t)/M_{\lambda'}(0)]/\ln[M_\lambda(t)/M_\lambda(0)]$ of logarithms of moments is independent of t if the corresponding $q(\lambda), q(\lambda')$ are those of the eigenvectors driving the physical phenomenon. Hence, different event spectra, corresponding to different freeze-out times, should generate the same ratio. Once a systematic set of such exponents $q(\lambda)$ have been identified, they give a strong hint of a binary multifragmentation mechanism and an estimate of b.

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